

Nonadiabatic Geometric Phase for the Cyclic Evolution of a Time-Dependent Hamiltonian System

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Abstract

The geometric phases of the cyclic states of a generalized harmonic oscillator with nonadiabatic time-periodic parameters are discussed in the framework of squeezed state. It is shown that the cyclic and quasicyclic squeezed states correspond to the periodic and quasiperiodic solutions of an effective Hamiltonian defined on an extended phase space, respectively. The geometric phase of the cyclic squeezed state is found to be a phase-space area swept out by a periodic orbit. Furthermore, a class of cyclic states are expressed as a superposition of an infinite number of squeezed states. Their geometric phases are found to be independent of \hbar , and equal to $-(n + 1/2)$ times the classical nonadiabatic Hannay angle.

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I. INTRODUCTION

Berry phase is one of the most important findings on the fundamental problem of quantum mechanics in recent years [1]. It reveals the gauge structure associated with a phase shift in adiabatic processes in quantum mechanics. This quantal phase is connected with a classical angle, namely, Hannay angle, by a simple and elegant expression in the semiclassical limit [2]. The next important progress has been the relaxation of the adiabatic approximation [3,4]. Aharanov and Anandan studied the phase associated with a cyclic evolution in quantum mechanics, which occurs when a state returns to its initial condition. They shown that the phase is a geometrical property of the curve in the projective Hilbert space which is naturally associated with the motion[3]. For the special case of an adiabatic evolution, this phase factor is a gauge-invariant generalization of the Berry phase. The significance of Aharanov and Anandan's generalization lies in: 1) The cyclic evolution of a physical system is of most interest in physics both experimentally and theoretically; 2) The universal existence of the cyclic evolution is guaranteed for any quantum system. The second point can be easily recognized by considering the eigenvectors of the unitary evolution operator for a quantum system. An explicit example is a time-periodic Hamiltonian system where the Floquet theorem applies. The eigenfunctions of the Floquet operator, which are so-called the Bloch wave functions in the condensed matter physics, are obviously cyclic solutions and of great interest in physics. However, unlike the adiabatic case, in the nonadiabatic case, calculating the eigenvectors and extracting the nonadiabatic geometric phase from the quasi-energy term for a time-dependent Hamiltonian is far from an trivial work, except for such a special example as the spin particle in magnetic field. Recent works of Ge and Child [5] made a step further in this direction. They found a *special* cyclic state of Gaussian wave packet's form for a generalized harmonic oscillator. The nonadiabatic geometric phase is explicitly calculated and found to be one half of the classical nonadiabatic Hannay angle.

In this paper, we would like to suggest an alternative approach - the squeezed state approach to study the *general* cyclic evolutions of the generalized harmonic oscillator. In particular, we shall construct a class of quantum states based on the superposition of an infinite number of squeezed states. We find that the condition for them to be cyclic evolutions is nothing but quantization rule without Maslov-Morse correction. The nonadiabatic geometric phases are obtained *analytically*, and found to be related to the classical Hannay angle by an explicit expression.

Squeezed state approach has wide applications in many branches of physics such as quantum optics and high energy physics etc. Recent years has witnessed a growing application of squeezed state to the study of chaotic dynamical systems [6–9]. In this paper we employ this approach to study the geometric phase and Hannay angle for the generalized harmonic oscillator. An apparent reason for this choice is that this system admits the squeezed state as an exact solution, and therefore is an ideal system for the study of time-dependent evolution of a squeezed state. First, we will introduce the squeezed state approach or squeezed state dynamics of the system. In this framework, the time-evolution of a squeezed state can be described by an effective Hamiltonian defined on an extended phase space and a differential equation describing the phase change. The periodic and quasiperiodic solution of the effective Hamiltonian correspond to the cyclic and quasicyclic evolution of a squeezed state, respectively. This enables us to study the cyclic evolution as well as the quasicyclic evolution

of a squeezed state, and their phase changes during the process. In particular, we focus on a class of cyclic states which are eigenstates of the Floquet operator and of special interest in quantum physics. The cyclic states can be expressed as a superposition of an infinite number of the squeezed states. Then, their geometric phases are obtained analytically. It is demonstrated that the quantal nonadiabatic geometric phase is connected with the nonadiabatic Hannay angle by a factor of $n + 1/2$. In light of our discussions, the quantal phase is found to be an area in the extended phase space swept out by a periodic orbit, in both adiabatic and nonadiabatic cases. This gives us a simple picture of the geometric meaning of the quantal phase.

The paper is organized as follows. In Sec. II, the squeezed state dynamics is introduced for a generalized harmonic oscillator. An effective Hamiltonian function and a differential equation describing the phase change during the evolution will be given. Then, in subsequent three sections, we are restricted to a specific choice of the periodic parameters. We shall derive the non-adiabatic Hannay angle analytically through the Lie transformation method in Sec. III. Sec. IV is devoted to the discussions on the cyclic or quasicyclic motion of a squeezed state and the phase change during the process. In Sec. V, we study the general cyclic states - the eigenstates of the Floquet operator, and show the connection between the geometric phase and the Hannay angle. In Sec VI, our discussions are generalized to a rather generic situation without referring to the specific form of the parameters. The paper concludes in Sec VII by discussions.

II. SQUEEZED STATE DYNAMICS OF A GENERALIZED HARMONIC OSCILLATOR

The squeezed state approach [10,11] starts from the time-dependent variational principle (TDVP) formulation,

$$\delta \int dt \langle \Phi, t | i\hbar \frac{\partial}{\partial t} - \hat{H} | \Phi, t \rangle = 0. \quad (1)$$

Variation w.r.t $\langle \Phi, t |$ and $|\Phi, t \rangle$ gives rise to the Schrödinger equation and its complex conjugate, respectively. In the squeezed state approach, the squeezed state is chosen as the trial wave function. The squeezed state is defined by the ordinary harmonic oscillator displacement operator acting on a squeezed vacuum state $|0\rangle$:

$$\begin{aligned} |\Psi\rangle &= \exp(\alpha \hat{a}^+ - \alpha^* \hat{a}) |\phi\rangle, \\ |\phi\rangle &= \exp\left(\frac{1}{2}(\beta \hat{a}^{+2} - \beta^* \hat{a}^2)\right) |0\rangle. \end{aligned} \quad (2)$$

\hat{a}^+ and \hat{a} are boson creation and annihilation operator which satisfy the canonical commutation relation: $[\hat{a}, \hat{a}^+] = 1$. Squeezed states are of great interest, offering many possible applications in diverse fields of quantum physics. Unlike coherent states, which have equal fluctuations in two directions of the phase space and minimize the uncertainty product of Heisenberg's uncertainty relation, the squeezed states have less fluctuation in one direction at the expense of increasing fluctuation in the other.

We define the coordinate and the momentum operators as:

$$\begin{aligned}
\hat{p} &= i\sqrt{\frac{\hbar}{2}}(\hat{a}^+ - \hat{a}), \\
\hat{q} &= \sqrt{\frac{\hbar}{2}}(\hat{a}^+ + \hat{a}).
\end{aligned} \tag{3}$$

Thus we have

$$\begin{aligned}
p &\equiv \langle \Psi, t | \hat{p} | \Psi, t \rangle, \\
q &\equiv \langle \Psi, t | \hat{q} | \Psi, t \rangle, \\
\Delta p^2 &\equiv \langle \Psi, t | (\hat{p} - p)^2 | \Psi, t \rangle = \hbar \left(\frac{1}{4G} + 4\Pi^2 G \right), \\
\Delta q^2 &\equiv \langle \Psi, t | (\hat{q} - q)^2 | \Psi, t \rangle = \hbar G, \\
\langle \Psi, t | \hat{q}\hat{p} + \hat{p}\hat{q} | \Psi, t \rangle &= 2qp + 4\hbar G\Pi.
\end{aligned} \tag{4}$$

The canonical coordinate (G, Π) was introduced by Jakiw and Kerman for the quantum fluctuation, and its relations with β in Eq.(2) is given in Ref. [12]. In fact, the squeezed state $|\Psi, t\rangle$ is equivalent to the following Gaussian-type state,

$$|\Psi, t\rangle = \frac{1}{(2G)^{1/4}} \exp\left(\frac{i}{\hbar}(p\hat{q} - q\hat{p})\right) \exp\left(\frac{1}{2\hbar}\left(1 - \frac{1}{2G} + 2i\Pi\right)\hat{q}^2\right) |0\rangle. \tag{5}$$

From the TDVP, we can obtain the dynamical equations for the expectation values and the quantum fluctuations,

$$\begin{aligned}
\dot{q} &= \frac{\partial H_{eff}}{\partial p}, & \dot{p} &= -\frac{\partial H_{eff}}{\partial q}, \\
\hbar\dot{G} &= \frac{\partial H_{eff}}{\partial \Pi}, & \hbar\dot{\Pi} &= -\frac{\partial H_{eff}}{\partial G},
\end{aligned} \tag{6}$$

where the dot denotes the time derivative. The effective Hamiltonian H_{eff} is defined on the extended space (q, p, G, Π) ,

$$H_{eff} = \langle \Psi, t | \hat{H} | \Psi, t \rangle. \tag{7}$$

These equations give us a simple and clear picture about the motion of the expectation values as well as the evolution of the quantum fluctuations.

The time-dependent variational principle leaves an ambiguity of a time-dependent phase $\lambda(t)$. When the trial state $|\Psi, t\rangle$ is transformed to $|\bar{\Psi}, t\rangle = e^{i\lambda(t)}|\Psi, t\rangle$, the derived variational equation of motion remains invariant. Therefore, we should fix the phase $\lambda(t)$ with the help of the Schrödinger equation,

$$\dot{\lambda}(t) = \langle \Psi, t | i\frac{\partial}{\partial t} | \Psi, t \rangle - \frac{1}{\hbar} \langle \Psi, t | \hat{H} | \Psi, t \rangle. \tag{8}$$

This phase is well defined for general *nonadiabatic* and *noncyclic* evolution of a squeezed state. It represents a phase change during a time-evolution of squeezed state. Obviously, the phase consists of two parts. The meaning of the second part is clear, it measures the time evolution. It is nothing but the *dynamical* phase and can be rewritten as,

$$\lambda_D(t) = -\frac{1}{\hbar} \int_0^t H_{eff} dt. \quad (9)$$

The first part can be viewed as a difference between the *total* phase and the *dynamical* phase. We call it *geometric* phase since it just is the Aharanov-Anandan's phase for the case of cyclic evolution. From the expression of the squeezed state, the geometric phase is equal to

$$\lambda_G(t) = \int_0^t \left(\frac{1}{2\hbar} (p\dot{q} - q\dot{p}) - \dot{\Pi}G \right) dt. \quad (10)$$

It is easy to see from the above formula that the evolution of expectation value (q, p) as well as the evolution of the quantum fluctuation (G, Π) contribute to the geometric phase. The contribution from the former one is \hbar dependent, while the contribution from quantum fluctuation is \hbar independent. For the case of cyclic squeezed state, the quantal phase is equal to a sum of the projective areas on the coordinates plane (q, p) and fluctuation plane (G, Π) swept out by a periodic orbit of the effective Hamiltonian.

In general situation, the squeezed state approach only gives an approximate solution to the Schrödinger equation since the trial wave functions are confined in a subspace of the Hilbert space. However, for a Hamiltonian containing only quadratic terms, such as the generalized harmonic oscillator that we shall discuss in the follows, it is easy to prove that the method is accurate since the system admits squeezed state as an exact solution.

The Hamiltonian of the generalized harmonic oscillator takes the form,

$$\hat{H}(q, p, t) = \frac{1}{2} \left(a(t)\hat{q}^2 + b(t)\hat{p}^2 + c(t)(\hat{q}\hat{p} + \hat{p}\hat{q}) \right), \quad (11)$$

where (\hat{q}, \hat{p}) are the quantum operators corresponding to the (q, p) , the conjugate variables of the phase space, and the real parameter $a(t), b(t), c(t)$ are periodic functions of the time with common period T .

Applying the squeezed state to this system, from Eq.(4) one can readily obtain an effective Hamiltonian in the extended phase space $(q, p; G, \Pi)$,

$$H_{eff}(q, p; G, \Pi; t) = H_{cl}(q, p, t) + \hbar H_{fl}(G, \Pi, t), \quad (12)$$

where

$$H_{cl} = \frac{1}{2} \left(a(t)q^2 + b(t)p^2 + 2c(t)qp \right), \quad (13)$$

describes the motion of the expectation values, and

$$H_{fl} = \frac{1}{2} \left(a(t)G + b(t)\left(\frac{1}{4G} + 4\Pi^2G\right) + 4c(t)G\Pi \right). \quad (14)$$

depicts the evolution of the quantum fluctuations.

III. NON-ADIABATIC HANNAY ANGLE

For the sake of simplicity, we first consider a specific choice of the periodic parameters, namely, $a(t) = 1 + \epsilon \cos(\omega t)$, $b(t) = 1 - \epsilon \cos(\omega t)$, $c(t) = \epsilon \sin(\omega t)$. Our discussions are restricted to the elliptic case, namely, $a(t)b(t) > c^2(t)$, i.e. $\epsilon < 1$. For one period evolution the parameters experience a circuit like $a + b = 2$; $(a - 1)^2 + c^2 = \epsilon^2$ in the parameter space. We would like to point out that the results given below is by no means limited to the specific chosen parameter form. As we shall show in Sec. VI that, the main discussions can be extended to the generic case without referring to the form of parameters $a(t), b(t), c(t)$.

The classical version of the generalized harmonic oscillator is described by a classical Hamiltonian of the form H_{cl} . Without the perturbation ($\epsilon = 0$), the phase plane of the system is full of the periodic orbits with period $T_0 = 2\pi$. The $(q, p) = (0, 0)$ is the unique fixed point. When the perturbation is added ($1 > \epsilon > 0$), the $q = p = 0$ remains a fixed point, however, all periodic orbits turn out to be quasi-periodic. In the Poincare section of $t = 2n\pi/\omega$ (n is an integer), one observes a family of invariant tori. These tori are driven by the Hamiltonian flow in such a way as to return to the original ones after the time $T = 2\pi/\omega$. This return leads to a classical angle change, which is naturally separated into two parts: a dynamical part and a geometric part. The later one is connected with a phase-space area swept out over time T and is named the non-adiabatic classical Hannay angle [4].

The classical Hamiltonian can be rewritten in terms of the action-angle variables, namely, $q = \sqrt{2I} \sin \phi$, $p = \sqrt{2I} \cos \phi$,

$$H_{cl} = H_0(I) + \epsilon H_1(I, \phi). \quad (15)$$

where $H_0 = I$, $H_1 = -I \cos(\omega t + 2\phi)$. It is convenient to employ the Lie transformation [13] method to make a canonical transformation, so that the new Hamiltonian $\bar{H}(\bar{I})$ contains the action variable only. Both the new Hamiltonian \bar{H} and the generating function w are expanded in the power series as $\bar{H} = \sum_{n=0}^{\infty} \epsilon^n \bar{H}_n$ and $w = \sum_{n=0}^{\infty} \epsilon^n w_n$. For simplicity we shall introduce some symbols first. Let $[\cdot, \cdot]$ represents a Poisson Bracket; Lie operator \mathcal{L}_n is then defined by $\mathcal{L}_n = [w_n, \cdot]$; and operator $\mathcal{D}_0 = \frac{\partial}{\partial t} + [\cdot, H_0]$. Inserting the series expansions and equating the terms of the same order of ϵ , one obtains a relation between the old and new Hamiltonian functions (in what follows, our solutions are accurate to ϵ^2),

$$\begin{aligned} \mathcal{D}_0 w_1 &= \bar{H}_1 - H_1, \\ \mathcal{D}_0 w_2 &= 2(\bar{H}_2 - H_2) - \mathcal{L}_1(\bar{H}_1 + H_1). \end{aligned} \quad (16)$$

To the first order, we choose \bar{H}_1 to eliminate secularities in the generating function w_1 , and then solve for the generating function w_1 . To the second order, we substitute w_1 into the right hand side, choose the \bar{H}_2 to eliminate secularities in w_2 , and solve for w_2 . Finally, we have the new Hamiltonian function in the form

$$\bar{H}(\bar{I}) = \bar{I} - \frac{\bar{I}}{\omega + 2} \epsilon^2. \quad (17)$$

The generating functions are $w_1 = \frac{I \sin(\omega t + 2\phi)}{\omega + 2}$ and $w_2 = 0$, respectively. The relation between the old variables and the new variables is given by

$$(\phi, I) = \mathcal{T}^{-1}(\bar{\phi}, \bar{I}), \quad (18)$$

where the transformation operator $\mathcal{T}^{-1} = 1 + \epsilon \mathcal{L}_1 + \epsilon^2(\mathcal{L}_2/2 + \mathcal{L}_1^2/2)$. The above relations can be expressed explicitly,

$$\begin{aligned} \phi &= \bar{\phi} - \frac{\sin(\omega t + 2\bar{\phi})}{\omega + 2} \epsilon + \frac{\sin(2\omega t + 4\bar{\phi})}{2(\omega + 2)^2} \epsilon^2, \\ I &= \bar{I} + \frac{2\bar{I} \cos(\omega t + 2\bar{\phi})}{\omega + 2} \epsilon + \frac{2\bar{I}}{(\omega + 2)^2} \epsilon^2. \end{aligned} \quad (19)$$

For this canonical transformation is explicitly time dependent, the new Hamiltonian \bar{H} differs from the old one H_{cl} both in value and in functional form. Thus, we introduce a function A to measure the difference,

$$A(\bar{\phi}, \bar{I}, t) = \bar{H}(\bar{I}) - H_{cl}(\phi(\bar{\phi}, \bar{I}, t), I(\bar{\phi}, \bar{I}, t), t). \quad (20)$$

Therefore the classical non-adiabatic Hannay angle is

$$\Theta_H = \langle \int_0^T \frac{\partial A}{\partial \bar{I}} dt \rangle_{\bar{\phi}_0}, \quad (21)$$

where the bracket denotes averaging around the invariant torus, $\langle \cdots \rangle = \frac{1}{2\pi} \int_0^{2\pi} \cdots d\bar{\phi}_0$.

With the help of Eqs. (15), (17) and (19), we finally arrive at the expression of the classical angle analytically,

$$\Theta_H = \frac{2\pi\epsilon^2}{(\omega + 2)^2}. \quad (22)$$

Obviously, this classical non-adiabatic Hannay angle is independent of the action.

IV. CYCLIC AND QUASICYCLIC SQUEEZED STATES

At $\epsilon = 0$, as mentioned before, the system reduces to the usual harmonic oscillator. The effective Hamiltonian can be expressed in terms of the action-angle variables,

$$H_{eff}(\epsilon = 0) = I + 2\hbar(J + 1/2), \quad (23)$$

where,

$$I = \frac{1}{2\pi} \oint pdq, \quad J = \frac{1}{2\pi} \oint \Pi dG. \quad (24)$$

The transformations between (q, p) , (G, Π) and (I, ϕ) , (J, θ) take following forms respectively,

$$\begin{aligned} q &= \sqrt{2I} \sin \phi, \\ p &= \sqrt{2I} \cos \phi, \end{aligned} \quad (25)$$

and

$$\begin{aligned}
G &= 2J + \frac{1}{2} - \sqrt{2J(2J+1)} \cos \theta, \\
\Pi &= \frac{\frac{1}{2}\sqrt{2J(2J+1)} \sin \theta}{(2J + \frac{1}{2} - \sqrt{2J(2J+1)} \cos \theta)}.
\end{aligned} \tag{26}$$

One can clearly see that, the motions of the two degrees of freedom are degenerate. The phase space is full of the periodic orbits with period $T_0 = 2\pi$ in (q, p) and period $T_0/2$ in (G, Π) , respectively. It is also worthwhile pointing out that, $q = p = 0$, $G = 1/2$, $\Pi = 0$ is the unique fixed point of the system.

For the case of $\epsilon \neq 0$, the variables (G, Π) are still decoupled from the variables (q, p) . In the fluctuation space (G, Π) , all periodic orbits degenerate to the quasi-periodic ones, whereas the fixed point $G = 1/2$, $\Pi = 0$ bifurcates to a periodic orbit, whose expression can be easily derived from Eq.(6) by using the method of power-series expansion,

$$\begin{aligned}
G_p(t) &= \frac{1}{2} - \frac{\cos(\omega t)}{\omega + 2} \epsilon, \\
\Pi_p(t) &= -\frac{\sin(\omega t)}{\omega + 2} \epsilon.
\end{aligned} \tag{27}$$

Then $(q = 0, p = 0; G_p(t), \Pi_p(t))$ is the unique solution with period T for the effective Hamiltonian. It corresponds to a cyclic squeezed state, whose expectation values keep fixed at the zero point, while its fluctuations (the width of the wave packet) change periodically. For this cyclic solution, its phase change during one period can be evaluated by Eqs.(8-10),

$$\lambda_G = -\int_0^T \dot{\Pi}_p G_p dt = \int_0^T \Pi_p \dot{G}_p dt = -\frac{\pi \epsilon^2}{(\omega + 2)^2}. \tag{28}$$

$$\lambda_D = -\int_0^T dt H_{fl}(G_p(t), \Pi_p(t)) = -\frac{1}{2} \left(1 + \left(-\frac{2}{(\omega + 2)} + \frac{2}{(\omega + 2)^2} \right) \epsilon^2 \right) T. \tag{29}$$

For the expectation values of the squeezed state keep fixed during the cyclic evolution, the geometric phase is independent of the Planck constant \hbar . Its value is equal to one half of the non-adiabatic Hannay angle (Eq.(22)) except for a negative sign.

Eq. (28) gives a unified expression for the Berry phase (in the adiabatic limit $\omega \rightarrow 0$) and the Aharonov and Anandan's phase (in nonadiabatic case). The geometric property of the former one can be readily understood from its independent of the time history. Whereas the geometric explanation of the nonadiabatic geometric phase has to resort to the projective Hilbert space. Our expression Eq. (28) of the geometric phase indicates a unified explanation, that is, the geometric property of the phase rests with its being an area surrounded by a periodic orbit in the fluctuation plane.

Now we turn our attention to a class of quasiperiodic motion in the extended phase space, that is, the fluctuations (G, Π) of a squeezed state keep staying on the periodic orbit $(G_p(t), \Pi_p(t))$, while its expectation value moves along a quasiperiodic orbit. With the help of the Hamiltonian (17), this quasiperiodic solution is described by

$$\bar{I}_q(t) = \bar{I}_0,$$

$$\bar{\phi}_q(t) = \bar{\phi}_0 + (1 - \frac{\epsilon^2}{\omega + 2})t. \quad (30)$$

(In all discussions, the initial time is set to be zero for convenience.)

Thus the geometric phase change is obtained from Eq.(10)

$$\lambda_G(t) = \frac{1}{\hbar} \int_0^t I(t) \dot{\phi}(t) dt + \int_0^t \Pi_p(t) \dot{G}_p(t) dt. \quad (31)$$

where (from Eqs.(19) and (30))

$$\begin{aligned} I(t) \dot{\phi}(t) &= I(\bar{I}_q(t), \bar{\phi}_q(t), t) \dot{\phi}(\bar{\phi}_q(t), \bar{I}_q(t), t) \\ &= \bar{I}_0 - \frac{\bar{I}_0 \omega}{\omega + 2} \cos(\omega t + 2t + 2\bar{\phi}_0) \epsilon \\ &+ \left(-\frac{2\bar{I}_0}{\omega + 2} + \frac{\bar{I}_0 \sin(2\omega t + 4t + 4\bar{\phi}_0)}{\omega + 2} - \frac{\bar{I}_0 \cos(2\omega t + 4t + 4\bar{\phi}_0)}{\omega + 2} + \frac{2\bar{I}_0}{(\omega + 2)^2} \right) \epsilon^2. \end{aligned} \quad (32)$$

The dynamical phase is (from Eq.(9))

$$\lambda_D = -\frac{1}{\hbar} \int_0^t H_{cl} dt - \int_0^t H_{fl}(G_p(t), \Pi_p(t), t) dt. \quad (33)$$

where (from Eqs.(13), (19) and (30)),

$$H_{cl} = \bar{I}_0 - \frac{\bar{I}_0 \omega}{\omega + 2} \cos(\omega t + 2t + 2\bar{\phi}_0) \epsilon + \left(\frac{2\bar{I}_0}{(\omega + 2)^2} - \frac{2\bar{I}_0}{\omega + 2} \right) \epsilon^2. \quad (34)$$

V. THE GENERAL CYCLIC STATES: EIGENSTATES OF THE FLOQUET OPERATOR

For a time-periodic Hamiltonian system, the Floquet theory applies. A unitary time evolution operator referring to one period T , the so-called Floquet operator $\hat{U}(T)$ is worthy of consideration, since its eigenstates are obviously the cyclic states, and of great interest in physics. The evolution operator satisfies $\hat{U}(mT) = \hat{U}^m(T)$. One may ask: What is the form of the eigenstates? How about the geometrical and dynamical phase of these cyclic states? Is there any connection between these phases with the Hannay's angle? These questions will be answered in this section.

To make things simple, let us start by considering the case $\omega = 1/s$, s is a natural number. Now, we construct a state as a superposition of infinite number of squeezed states, i.e.

$$|S_1\rangle = c \int_0^{2\pi} e^{\frac{i}{\hbar} \bar{I}_0 \bar{\phi}_0} |\bar{I}_0, \bar{\phi}_0; G_0, \Pi_0\rangle d\bar{\phi}_0. \quad (35)$$

where $|\bar{I}_0, \bar{\phi}_0; G_0, \Pi_0\rangle$ represents a squeezed state centered at $q(\bar{I}_0, \bar{\phi}_0, t=0), p(\bar{I}_0, \bar{\phi}_0, t=0)$ (see Eqs.(19) and (25)) with fluctuations G_0, Π_0 ; The G_0, Π_0 are chosen on the unique periodic orbit ($G_0 = G_p(t=0), \Pi_0 = \Pi_p(t=0)$) (see Eq.(27)); c is a normalization constant.

Acting the Floquet operator on the state, we have from Eqs. (30), (31) and (33),

$$\hat{U}(T)|S_1\rangle = c \int_0^{2\pi} e^{\frac{i}{\hbar}\bar{I}_0\bar{\phi}_0} e^{i(\lambda_G(T)+\lambda_D(T))} |\bar{I}_0, \bar{\phi}_0 + (1 - \frac{\epsilon^2}{\omega+2})T; G_0, \Pi_0\rangle d\bar{\phi}_0. \quad (36)$$

From Eqs.(31) and (33), we obtain the expression of the phases,

$$\lambda_G(T) = \frac{\bar{I}_0 T}{\hbar} \left(1 - \frac{2\epsilon^2}{\omega+2} + \frac{2\epsilon^2}{(\omega+2)^2} \right) - \frac{\pi\epsilon^2}{(\omega+2)^2}. \quad (37)$$

and

$$\lambda_D(T) = \lambda_D^R = - \left(\frac{\bar{I}_0}{\hbar} + \frac{1}{2} \right) \left(1 + \left(\frac{-2}{\omega+2} + \frac{2}{(\omega+2)^2} \right) \epsilon^2 \right) T. \quad (38)$$

The minimum period of the trigonometry functions in the expression (32) and (34) is $\frac{2s\pi}{2s+1}$, integration period in the calculation of the phases is $2s\pi$. Therefore, the terms containing $\bar{\phi}_0$ vanish.

The geometric phase is divided into following two parts,

$$\lambda_G(T) = \frac{\bar{I}_0 T}{\hbar} \left(1 - \frac{\epsilon^2}{\omega+2} \right) + \lambda_G^R. \quad (39)$$

where

$$\lambda_G^R = - \left(\frac{\bar{I}_0}{\hbar} + \frac{1}{2} \right) \frac{2\pi\epsilon^2}{(\omega+2)^2}. \quad (40)$$

As we will see later, the first part will compensate for a phase change caused by the displacement of the expectation value, which makes the integrand having the same form as the original one under a variable transformation.

Making variable transformation $\bar{\phi}'_0 = \bar{\phi}_0 + (1 - \frac{\epsilon^2}{\omega+2})T$, we have

$$\hat{U}(T)|S_1\rangle = c e^{i(\lambda_D^R + \lambda_G^R)} \int_{(1-\frac{\epsilon^2}{\omega+2})T}^{2\pi+(1-\frac{\epsilon^2}{\omega+2})T} e^{\frac{i}{\hbar}\bar{I}_0\bar{\phi}'_0} |\bar{I}_0, \bar{\phi}'_0, G_0; \Pi_0\rangle d\bar{\phi}'_0. \quad (41)$$

The integral in above formula can be divided into three parts,

$$\int_{(1-\frac{\epsilon^2}{\omega+2})T}^{2\pi+(1-\frac{\epsilon^2}{\omega+2})T} \dots = \int_0^{2\pi} \dots + \int_{2\pi}^{2\pi+(1-\frac{\epsilon^2}{\omega+2})T} \dots - \int_0^{(1-\frac{\epsilon^2}{\omega+2})T} \dots. \quad (42)$$

The last two terms will cancel each other if and only if $e^{\frac{i}{\hbar}\bar{I}_0 2\pi} = 1$, which gives the condition for the state $|S_1\rangle$ being a cyclic state,

$$\bar{I}_0 = n\hbar. \quad (43)$$

This is nothing but the quantization rule without Maslov-Morse correction.

Under this condition, we get

$$\hat{U}(T)|S_1\rangle = e^{i(\lambda_D^R + \lambda_G^R)}|S_1\rangle. \quad (44)$$

Actually, the state $|S_1\rangle$ is an eigenstate of the Floquet operator, n is the state number. In comparison with Eq.(22), we finally reach a simple relation between the geometrical phase and the non-adiabatic Hannay angle,

$$\lambda_G^R = -(n + \frac{1}{2})\Theta_H. \quad (45)$$

Now we extend the above discussions to the general case, i.e. $\omega = r/s$, where r, s are co-primed natural numbers. First, let us consider the situation that $\hat{U}(mT)$ acts on the state $|S_1\rangle$,

$$\hat{U}(mT)|S_1\rangle = c \int_0^{2\pi} e^{\frac{i}{\hbar}\bar{I}_0\bar{\phi}_0} e^{i\lambda} |\bar{I}_0, \bar{\phi}_0 + (1 - \frac{\epsilon^2}{\omega + 2})mT; G_0, \Pi_0\rangle d\bar{\phi}_0. \quad (46)$$

Making variable transformation $\bar{\phi}'_0 = \bar{\phi}_0 + (1 - \frac{\epsilon^2}{\omega + 2})mT$ and using the condition Eq.(43) one has,

$$\hat{U}(mT)|S_1\rangle = c \int_0^{2\pi} e^{\frac{i}{\hbar}\bar{I}_0\bar{\phi}'_0} e^{i\lambda'} |\bar{I}_0, \bar{\phi}'_0; G_0, \Pi_0\rangle d\bar{\phi}'_0. \quad (47)$$

where the phase λ' can be naturally divided into two parts $\lambda' = \lambda_m^1 + \lambda_m^2(\bar{\phi}'_0)$.

The first term does not contain the variable $\bar{\phi}'_0$; Those trigonometry functions in the phase that relates to the $\bar{\phi}'_0$ are included in the second term. Then we can derive the expression of the first term through a simple analysis,

$$\lambda_m^1 = m(\lambda_G^R + \lambda_D^R). \quad (48)$$

Now we construct a new state,

$$|S_r\rangle = |S_1\rangle + \dots + e^{-i\lambda_m^1} \hat{U}(mT)|S_1\rangle + \dots e^{-i\lambda_{r-1}^1} \hat{U}((r-1)T)|S_1\rangle. \quad (49)$$

Acting the Floquet operator on this state, we get,

$$\hat{U}(T)|S_r\rangle = \hat{U}(T)|S_1\rangle + \dots + e^{-i\lambda_m^1} \hat{U}((m+1)T)|S_1\rangle + \dots + e^{-i\lambda_{r-1}^1} \hat{U}(rT)|S_1\rangle. \quad (50)$$

Since

$$\hat{U}(rT)|S_1\rangle = c \int_0^{2\pi} e^{\frac{i}{\hbar}\bar{I}_0\bar{\phi}'_0} e^{i(\lambda_r^1 + \lambda_r^2(\bar{\phi}'_0))} |\bar{I}_0, \bar{\phi}'_0; G_0, \Pi_0\rangle d\bar{\phi}'_0. \quad (51)$$

From Eqs.(32) and (34), we know that, the minimum period of those trigonometry functions containing $\bar{\phi}_0$ is $\frac{2\pi s}{2s+r}$, whereas the integral interval in calculating phase is $rT = 2\pi s$. So the second term containing the $\bar{\phi}_0$ in the phase vanishes, and the last term in the Eq.(49) becomes,

$$e^{-i\lambda_{r-1}^1} \hat{U}(rT)|S_1\rangle = e^{i(\lambda_r^1 - \lambda_{r-1}^1)}|S_1\rangle. \quad (52)$$

From relation (48), one has $\lambda_m^1 - \lambda_{m-1}^1 = \lambda_G^R + \lambda_D^R$ for any m . Substituting this relation into Eq.(52) and then into Eq.(50) yields

$$\hat{U}(T)|S_r\rangle = e^{i(\lambda_D^R + \lambda_C^R)}|S_r\rangle. \quad (53)$$

Then we conclude that the relation (45) still holds for the case $\omega = r/s$. As ω is an irrational number, we can use a series of rational numbers to approach it. So we are convinced to argue that, for any ω the geometric phase of the eigenvector of the Floquet operator is equal to $-(n + 1/2)$ times the Hannay angle.

The expression of the geometric phase λ_G^R (see Eq.(40)) consists of two parts. As discussed in Sec.IV, the geometric meaning of the second term rests with its representing an area in the fluctuation plane (G, Π) swept out by a periodic orbit. What is the meaning of the first term? Now we should return to the coordinate plane (q, p) and consider this problem in the action-angle variables (I, ϕ) . Let us consider following differential 2-form which is preserved under the canonical transformation $(I, \phi) \rightarrow (\bar{I}, \bar{\phi})$,

$$dI \wedge d\phi - dH \wedge dt = d\bar{I} \wedge d\bar{\phi} - d\bar{H} \wedge dt. \quad (54)$$

Then the above formula can be rewritten as

$$dI \wedge d\phi = d\bar{I} \wedge d\bar{\phi} - d(\bar{H} - H) \wedge dt. \quad (55)$$

Making an integration of the above equation for one period (T) , and comparing it with Eqs.(31) and (39), one will find immediately that the geometric phase corresponds to the second term on the right hand side of the above equation. The areas in the phase plane (I, ϕ) and $(\bar{I}, \bar{\phi})$ can be expressed respectively as,

$$a_1 = \int_0^T I \dot{\phi} dt, \quad a_2 = \int_0^T \bar{I} \dot{\bar{\phi}} dt. \quad (56)$$

Keeping in mind that the area meaning of the differential 2-form, one will find that the first term of the geometric phase λ_G^R represents the difference of the area through a canonical transformation. That is $\langle a_1 - a_2 \rangle_{\phi_0} / \hbar$. Of course, an averaging over the angle variable should be made.

VI. EXTENSION TO GENERAL SITUATION

In the preceding sections, our discussions are restricted to a specific choice of the periodic parameters $a(t), b(t)$ and $c(t)$. Explicit perturbative results are obtained for the geometric phase and the Hannay angle. In this section, we shall demonstrate that the main results obtained in previous sections are also true for a generic situation. The similar discussions can be done regardless of any concrete form of the periodic parameters.

In fact, whatever the form $a(t), b(t)$ and $c(t)$ have, under the elliptic condition $a(t)b(t) > c^2(t)$, the phase plane of the expectation values (q, p) and the fluctuations (G, Π) have the same topological structure as that discussed in Secs. III and IV. One can clearly see that, in the fluctuation plane (G, Π) , all motions are restricted on the invariant tori except for a unique T-periodic solution $(G_p(t), \Pi_p(t))$. Whereas, for the Hamiltonian system H_{cl} , the $(q = 0, p = 0)$ is obviously a fixed point, other motions are quasi-periodic trajectories confined on the invariant tori. This similarity in the topology of the phase structure provides a basis for our generalization as given in the follows.

Through a canonical transformation, $q = q(\bar{I}, \bar{\phi}, t)$, $p = p(\bar{I}, \bar{\phi}, t)$, or inversely, $\bar{I} = \bar{I}(q, p, t)$, $\bar{\phi} = \bar{\phi}(q, p, t)$, the Hamiltonian $H_{cl}(q, p, t)$ can be transformed to a new Hamiltonian $\bar{H}(\bar{I}, t)$ which does not contain the angle variable $\bar{\phi}$. We still choose a state $|S_1\rangle$ in the same form as in Eq.(35), and consider the situation that $\hat{U}(mT)$ acts on the state $|S_1\rangle$,

$$\hat{U}(mT)|S_1\rangle = c \int_0^{2\pi} e^{\frac{i}{\hbar} \bar{I}_0 \bar{\phi}_0} e^{i\lambda} |\bar{I}_0, \bar{\phi}_0 + \int_0^{mT} \frac{\partial \bar{H}(\bar{I}_0, t)}{\partial \bar{I}_0} dt; G_0, \Pi_0\rangle d\bar{\phi}_0. \quad (57)$$

where

$$\lambda = \lambda_D(mT) + \lambda_G(mT) \quad (58)$$

$$\lambda_D(mT) = -\frac{1}{\hbar} \int_0^{mT} H_{eff} dt \quad (59)$$

$$\lambda_G(mT) = \frac{1}{\hbar} \int_0^{mT} \frac{1}{2} (p\dot{q} - q\dot{p}) dt - \int_0^{mT} \dot{\Pi}_p G_p dt \quad (60)$$

$$\lambda_D(mT) = \langle \lambda_D(mT) \rangle_{\bar{\phi}_0} + \{ \lambda_D(mT) \}(\bar{\phi}_0),$$

$$\lambda_G(mT) = \langle \lambda_G(mT) \rangle_{\bar{\phi}_0} + \{ \lambda_G(mT) \}(\bar{\phi}_0). \quad (61)$$

where the symbols $\langle \dots \rangle_{\bar{\phi}_0}$ denotes the average over $\bar{\phi}_0$ as in Eq.(21); $\{ \dots \}(\bar{\phi}_0)$ represent the terms relating to $\bar{\phi}_0$.

$$\langle \lambda_G(mT) \rangle_{\bar{\phi}_0} = \frac{m}{\hbar} \langle \int_0^T \frac{1}{2} (p\dot{q} - q\dot{p}) dt \rangle_{\bar{\phi}_0} - m \oint G_p d\Pi_p. \quad (62)$$

Making variables transformation $\bar{\phi}'_0 = \bar{\phi}_0 + \int_0^{mT} \frac{\partial \bar{H}}{\partial \bar{I}_0} dt$ and under the condition $\bar{I}_0 = n\hbar$, we have

$$\hat{U}(mT)|S_1\rangle = ce^{i\lambda_m^1} \int_0^{2\pi} e^{\frac{i}{\hbar} \bar{I}_0 \bar{\phi}'_0} e^{i\{ \lambda_D(mT) \}(\bar{\phi}'_0) + \{ \lambda_G(mT) \}(\bar{\phi}'_0)} |\bar{I}_0, \bar{\phi}'_0; G_0, \Pi_0\rangle d\bar{\phi}'_0 \quad (63)$$

where

$$\lambda_m^1 = m(\lambda_G^R + \lambda_D^R). \quad (64)$$

and the geometric and dynamical parts take the forms as follows,

$$\lambda_G^R = \frac{1}{\hbar} \left(\langle \int_0^T \frac{1}{2} (p\dot{q} - q\dot{p}) dt \rangle_{\bar{\phi}_0} - \bar{I}_0 \int_0^T \frac{\partial \bar{H}}{\partial \bar{I}_0} dt \right) - \oint G_p d\Pi_p. \quad (65)$$

$$\lambda_D^R = -\frac{1}{\hbar} \langle \int_0^T H_{eff} dt \rangle_{\bar{\phi}_0} \quad (66)$$

The motion of the expectation values (q, p) confined on the invariant torus \bar{I}_0 is quasi-periodic. Ergodicity of the motion guarantees that the temporal average is equivalent to

the spatial average supposing that the time is long enough. Then, in light of the ergodicity principle we can choose an integer r , which is large enough so that the terms containing $\bar{\phi}_0$ in phase expression λ in Eq.(57) almost vanish. Then, similar to Eq. (49), we construct a state $|S_r\rangle$ and readily prove that the relation (53) still holds. Now let us see the meaning of the geometric phase λ_G^R expressed by (65). Similar to (54) and (55), we consider the following differential 2-form which is preserved under the canonical transformation, i.e.

$$dp \wedge dq - d\bar{I} \wedge d\bar{\phi} = -d(\bar{H} - H_{cl}) \wedge dt. \quad (67)$$

Similar to discussions in the last paragraph of the former section, let us first make an integration of the above equation for one period (T), then average over the variable $\bar{\phi}_0$. Keeping in mind that the area meaning of the differential 2-form, one will find immediately that the term bracketed in the expression of the geometric phase (65) corresponds to the left hand side of the above equation, whereas the right hand side will equal to $n\hbar$ times the Hannay's angle (refer to (21)). 1/2 relation between the first term in (65) and the classical angle is given by Ge and Child [5] and verified by our explicit perturbative results in former sections. Thus, we arrive at the same conclusion as is shown in Eq.(45).

VII. CONCLUSIONS AND DISCUSSIONS

In this paper, the nonadiabatic geometrical phase and the Hannay angle of the cyclic evolutions for a generalized harmonic oscillator are studied by using the squeezed state approach. It is demonstrated that the squeezed state approach is a very powerful method, which enables us to obtain analytically and explicitly the geometrical phase and the Hannay angle.

In the framework of the squeezed state, the time-dependent evolution of a squeezed state can be described by an effective Hamiltonian on an extended phase space and a differential equation describing the phase change. The periodic and quasiperiodic solution of the effective Hamiltonian system corresponds to the cyclic and quasicyclic evolution of a squeezed state, respectively. Then, we found a unique cyclic squeezed state, whose expectation value keeps staying at the zero point, while the fluctuations change periodically. We obtain a unified expression of its geometrical phase for the adiabatic as well as nonadiabatic case. The geometric property lies in its equality to an area on the extended phase space swept out by a periodic orbit.

A special class of cyclic states of the system are of great interest in physics. They are the so-called the Floquet states -eigenvectors of the Floquet operator. These cyclic states can be expressed as a superposition of an infinite number of the squeezed states. Their geometric phases are obtained analytically. The quantum phase can be interpreted as a sum of the area difference on the expectation value plane through the canonical transformation and the area on the quantum fluctuation plane swept out by a periodic orbit. This explanation provides a unified picture of the geometric meaning of the quantal phase for the adiabatic case as well as the nonadiabatic case.

We have also discussed the classical version of the system. The nonadiabatic Hannay angle is obtained by employing Lie transformation method, which is found to be independent of the action. The geometrical phases of those cyclic states are equal to $-(n+1/2)$ times the

Hannay angle. In the adiabatic limit, our $n + 1/2$ relation is identical to the elegant formula of Berry [2]. However, the semiclassical approximation has not been invoked. Therefore, we believe that, in addition to the semiclassical method, the squeezed state approach provides an alternative way to bridge the classical and quantum world.

An interesting example is given by $n = 0$, i.e. the ground state of the Floquet states. It corresponds to the unique cyclic squeezed state mentioned above. The geometrical phase of this cyclic state resulting only from the periodic evolution of the fluctuations' part, is equal to one half of the classical angle. This is just what obtained by Ge and Child [5].

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